Scaling and Estimation

As an additional material we prepared for you more scaling and estimation examples. The famous estimation examples were prepared by Enrico Fermi (Fermi problem - Wikipedia). You can find many of them on the Internet, probably the most cited is: How many piano tuners are there in Chicago?

When you are interested, you can find more examples on website and if you want you can look for competitions in solving such tasks in some countries.

Soutěž Fermiho úlohy: archiv minulých ročníků (upol.cz)

30 Fermi problems for students and teachers — Innovative Teaching Ideas | Teaching ResourcesFermihoProblemyJK.pdf (prase.cz)

Example 1.

Materials needed: Vernier calipers, graph paper and some oranges, grapefruits, or tangerines, beaker.

Procedure

Use calipers to measure the diameter of the fruits. Find two of them where one diameter is twice the other.

Using some beaker full of water, measure the volume of the selected fruits by overflowed water.

Carefully peel the selected fruit and arrange each of them in rectangular shape on the separate graph paper and so calculate the surface areas of the selected fruits.

Do the calculations signing the first fruit the bigger one:

$$\frac{Surface\ area\ of\ the\ first\ fruit}{Surface\ area\ of\ the\ second\ fruit}$$

$$\frac{Volume\ of\ the\ first\ fruit}{Volume\ of\ the\ second\ fruit}$$

$$\frac{Diameter\ of\ the\ first\ fruit}{Diameter\ of\ the\ second\ fruit} = scale\ factor\ (k)$$

Now you can see that the area increases by the scale factor $k.k = k^2$ and volume by the scale factor $k.k.k = k^3$

Example 2

Scaling and festivals.

Some people enjoy attending big festivals, while others prefer smaller ones. Let's compare a festival for 1000 people to another one for 10000 people, which is ten times larger. Is there a difference, or is it just more people?

Well, if we want to make sure everyone has the same amount of space and comfort, we need to make some adjustments:

First, we need to make the festival area bigger. Since we have ten times more people, the area should also be ten times larger. Imagine the festival area shaped like a rectangle. Each side of the rectangle needs to be about 3.16 times longer (that's the square root of 10). So, even though each person still has the same amount of space, they will need to walk around three times more during the festival!

While the festival area grows tenfold, the perimeter (the outer boundary) only increases by 3.16 times. This can create a problem. Buffets and stands are usually placed around the perimeter, but it won't be enough to accommodate everyone. That means some stands will have to be placed inside the area instead.

When you want to attract ten times more people, you might need to hire more famous musicians. These famous musicians often ask for a lot of money. As a result, big festivals may have more expensive tickets to cover the cost, even if more people attend.

But not everything changes with size. For example, the festival still lasts three days, regardless of whether it's big or small. At a small festival with fewer stages, you can manage to see more of the program because you have more time for each performance. However, at a big festival, you might only have a few minutes for each stage because there are so many performers.

So, in summary, there are some differences when it comes to big and small festivals. The area needs to be larger, the placement of stands may need to be adjusted, and bigger festivals may have more famous musicians and higher-priced tickets. However, the duration of the festival remains the same, and the time you get to spend at each stage might be different depending on the festival's size.

Idea from Samuel Kováčik
Škálovania festivalov - Vedátor (vedator.space)

Example 3.

Do you know the story from Jules Verne Around the World in 80 Days? How long will it take to the ant to orbit the Earth travelling along the equator? To solve this problem, we will make some estimations.

Watching an ant moving on a hand, which means probably 50 mm, let us measure it several times

Number of the measurement	Time in seconds
1.	1
2.	2
3.	2
4.	3
5.	2
6.	4
7.	1
8.	2
9.	2
10.	1

The average time is the sum of all times divided by the number of measurements (arithmetic mean). Which gives us $\frac{(1+2+2+3+2+4+1+2+2+1)s}{10} = 2s$.

The speed of ant is then $v = \frac{50\text{mm}}{2\text{s}} = \quad 25\frac{\text{mm}}{\text{s}}$ which is 0,025 m/s.

Estimation of the distance travelled: Radius of Earth – 10^7 m = 100 000 000 m

The distance along the equator: $2\pi R = 2x3{,}14x100\,000\,000m \cong 6.100\,000\,000m$ =6.10^8m

Necessary time would be: $\frac{6.10^8 m}{2{,}510^{-2}\frac{m}{s}} = 2{,}4.10^{10}s \approx 10^{10}s.$

When we do not have time for watching moving ant. We can use mathematical way of estimation.

We will *underestimate and overestimate the speed of an ant*: it is enough to watch an ant moving on a hand only ones. Our estimation will be then in the interval (1-4) seconds. We will use

Geometric mean:$\sqrt{1s.4s} = 2s.$

The speed of ant is then 25 mm/s, which is 0,025 m/s.

The rest is the same as above.

<u>Let's talk about the steps we followed to calculate the time it took for the ant to travel around the world.</u>

To figure out the time, we needed to know how fast the ant could move and how long the path was.

Since we didn't know the exact values, like the Earth's radius, we made estimations. We guessed values that were higher and lower than the actual ones, and then we found the average value. If we can measure things, we can make even better estimations.

Once we had the estimations, we calculated the time it would take for the ant to travel around the world.

We also used a useful skill called rounding. When we have big numbers, we can round them to the nearest power of ten. For example, if the number of inhabitants in Europe was 7,464,000,000, use 7 500 000 000 for better scientific notation $7,5.10^9$

Radius of Earth 6378 km rounded 6 000 000 m= 6.10^6 or we can even use only 10^7

<u>You might be wondering, why do we use estimations? Well, practicing estimations can help us in many ways:</u>

It helps us understand the problem without getting too caught up in precise calculations.

We can focus more on the scientific part of the problem instead of getting stuck in algebra and arithmetic.

Estimations allow us to break down a big problem into smaller parts.

It teaches us how to calculate things without always relying on calculators.

Estimations develop our common sense and give us a good sense of numbers.

By using estimations, we can solve problems like the ant's journey and learn important skills along the way. The more we practice, the better we get at it!

Now, there's something called the arithmetic mean and the geometric mean, which are different ways of finding averages. The arithmetic mean is the most common type of average, where we add numbers and divide them. The geometric mean involves multiplying numbers and finding the root of them. It's a more advanced concept, but for our ant's journey and other estimations, we don't need to worry about it.

<u>Example 4.</u>

How many people are calling or writing a message on their mobile phone right now?

Solution

We need to:

- estimate what fraction of time are people calling or writing on phone,
- round the number of people on Earth.

Time spent on the mobile phone in a day: $0{,}5\ hours \leq t \leq 8\ hours$

$$t = \sqrt{0{,}5\mathrm{h}\ \cdot 8\mathrm{h}} = 2\mathrm{h}$$

Calculate the fraction of time and round it to the power of ten:

$$\frac{2\mathrm{h}}{24\mathrm{h}} \cong 0{,}08\ \approx 10^{-2}$$

(This result is the probability an average person is calling or writing a message right now)

Number of people on Earth: $\approx 7.10^{9}$

Number of people calling or writing message on their mobile phone is:

$$7.10^{9}.10^{-2} = 7.10^{7}$$

Example 5.

The chance of winning the lottery is one in a hundred million. How tall would be the stack if we stacked the lots covering all the possibilities?

Solution

To solve the problem, we need to know two pieces of information: how many tickets there are and the thickness of one ticket.

If only one ticket wins, we have 10^8 options - tickets.

How thick is one ticket? Let us guess.

The paper package (500 pcs.) is about 5 cm, this will be our lower limit – 0,01 cm. However, the lottery tickets tend to be a bit thicker. Let us try cards -our upper limit. The deck of 32 cards is about 1 cm. This will be 0,03 cm for one ticket.

Geometric mean $\sqrt{0{,}01cm.\,0{,}03cm} = 0{,}017cm \cong 0{,}02cm$

The height of all tickets will be: $2.\,10^{-4}m.\,10^8 = 2.\,10^4m$

The airplanes fly usually in that altitude (because of convenient density of air).

Example 6.

Why could King Kong never exist?

Solution

According to Wikipedia King Kong has appeared in various media since 1933. His height varies from around 7m to 18m. What about his weight?

Weight is how heavy something is, and it's related to how big it is. It is proportional to the cube of size - h^3 (h is height).The bigger something is, the heavier it tends to be. Strength is how strong something is, like how much it can lift or carry. When something gets bigger, its strength increases too, but not as fast as its weight. I tis proportional to the square of size - h^2.

Let's imagine a gorilla that is 1.5 meters tall. If we slowly make it bigger, its weight and strength will both increase. However, the weight will increase more quickly than the strength. The weight goes up faster than the strength.

Eventually, when the gorilla gets to be around 2.5 to 3 meters tall, it will become too heavy for its own bones to support it, and it will collapse. This limit can vary for different animals.

So, animals can't keep growing forever without changing their shape. Also, they can't keep getting bigger forever anyway!

For example, elephants have really thick legs compared to their bodies because they need them to support their big size. Mice, on the other hand, have thin legs because they're small and don't need as much support.

<u>Example 7.</u>

What is more efficient to be big or small?

Solution

The energy received by a warm-blooded animal (*E*), which is supposed to maintain its temperature during time Δt, is probably proportional to its mass m, which in turn is proportional to the volume *V* of the animal. The volume of the body is directly proportional to the cube of length or height *l*.

Thus, we could write proportions for the energy: $E{\sim}m{\sim}V{\sim}l^3$

Now, let's think about how the animal loses some of that energy (ΔE) in time Δt to its surroundings. The energy that escapes from the animal's body is related to the surface area (*A*)of its body. Surface area of the animal's body surface will be directly proportional to the square l^2 of its linear dimensions.

So, it applies $\Delta E \sim A \sim l^2$.

If we now divide the energy that escaped in time Δt from the surface of the mammal into the environment by the energy needed to maintain body temperature we get:

$$\frac{\Delta E}{E} = \frac{S}{V} \Rightarrow \frac{\Delta E}{E} = \frac{l^2}{l^3} = \frac{1}{l}$$

If we compare the energy that escapes to the energy needed to maintain body temperature, we find that bigger animals lose a smaller portion of their energy through their body surface. This is because they have more volume inside their bodies, which helps them retain heat. On the other hand, smaller animals, like hamsters or mice, have a larger surface area compared to their volume. This means they lose heat more quickly, and they need more energy to stay warm.

That's why children, especially during growth periods, need more food and energy than adults. Even small people have a higher surface-to-volume ratio, so they lose heat faster. If you've ever had a hamster or a mouse, you might have noticed that they eat a lot because they need to keep their bodies warm.

So, in summary, bigger animals need more energy to stay warm, and smaller animals lose heat more quickly. That's why children need more food and energy during their growing years, and small creatures like hamsters or mice eat most part of the day.

If you want to find out more similar problems, there's a book called "Why Size Matters" by J.T. Bonner and another book called "A Mathematical Nature Walk" by J.A. Adam.